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Cohomology theories for complexes

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Abstract

We introduce and study a complete cohomology theory for complexes, which provides an extended version of Tate–Vogel cohomology in the setting of (arbitrary) complexes over associative rings. Moreover, for complexes of finite Gorenstein projective dimension a notion of relative Ext is introduced. On the basis of these cohomology groups, some homological invariants of modules over commutative noetherian local rings, such as Martsinkovsky's ξ-invariants and relative and Tate versions of Betti numbers, are extended to the framework of complexes with finite homology. The relation of these invariants with their prototypes is explored.

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1. Introduction

The purpose of this paper is to extend Tate–Vogel and relative cohomology theories to the setting of *unbounded* complexes of modules over associative rings.

Tate cohomology was created in the 1950s, based on Tate's observation that the $\mathbb{Z}G$-module $\mathbb{Z}$ with the trivial action admits a complete projective resolution [9, Ch. XII]. At first, it was defined for finite groups G equipped with a $\mathbb{Z}[G]$-module M. It was extended by Farrell [11] to discrete groups having finite virtual cohomological dimension and was further extended by Buchweitz [8] to two-sided noetherian Gorenstein rings. Later, Benson and Carlson [7], Mislin [18], and Vogel [13] independently developed a generalization of Tate cohomology applicable to all pairs of modules over associative rings which coincides with Tate cohomology when the underlying ring is Gorenstein. More recently, Tate cohomology for finite modules of finite Gorenstein dimension over noetherian rings has been studied explicitly by Avramov and Martsinkovsky [6].

Tate–Vogel cohomology has also been extended to the framework of complexes. Goichot [13, Sec. III] defined Vogel cohomology for complexes of modules over associative rings and a Tate cohomology theory for complexes of finite Gorenstein projective dimension was introduced by Veliche in [20, Sec. 4]; however (unlike the case for

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modules) if the complex in the first argument is of finite Gorenstein projective dimension Goichot's extension does not coincide with Veliche's extension.

Keeping this in mind, in the first part of the paper we introduce and study a Vogel cohomology theory for complexes which is compatible with the Veliche extension of Tate cohomology theory which appeared in [20], provided that the complex in the first argument is of finite Gorenstein projective dimension. It is shown that most of the properties of Vogel cohomology for modules extend well to the framework of complexes; for instance, we show that a homologically bounded above complex M is of finite projective dimension if and only if $\widetilde{\mathrm{Ext}}_R^0(M, M) = 0$ (see Proposition 3.1.3 below). This result extends a result of Kropholler [15, 4.2.4] to the setting of complexes. In a different approach, we introduce a notion of complete cohomology for complexes using semi-injective resolutions and then compare these two notions and show that their equivalence is related heavily to the finiteness of the two invariants, silp R, the supremum of the injective lengths of projectives, and spli R, the supremum of the projective lengths of the injectives [12].

The relative cohomology theory was introduced by Eilenberg and Moore in their 1965 AMS Memoir [10]. This theory was further studied by MacLane [16]. When R is a two-sided noetherian ring and (left) R-module M admits a *proper* resolution $G \to M$ by finite modules of Gorenstein dimension 0, Avramov and Martsinkovsky [6] associate with M, for each $n \in \mathbb{Z}$ and each R-module N, a relative cohomology group $\mathrm{Ext}^n_{\mathcal{G}}(M, N)$ by setting $\mathrm{Ext}^n_{\mathcal{G}}(M, N) = \mathrm{H}^n\mathrm{Hom}_R(G, N)$. Veliche [20, Sec. 6] generalized their results to the more general set-up when M admits a proper resolution by Gorenstein projective modules over an arbitrary ring R. In this paper we also aim to present an extended version of relative cohomology functors to the setting of complexes. When M is a complex of finite Gorenstein projective dimension and N is an R-module, we use cohomology groups of the total complex $(P_M, P_N)_{ba}$, to define a notion of relative Ext, denoted as $\mathrm{Ext}^*_{\mathcal{GP}}(M, N)$, where P_M (resp. P_N) is a semiprojective resolution of M (resp. N) and the subscript ba applied to the Hom functor (P_M, P_N) serves for bounded above morphisms. We show that these relative cohomology groups have the properties that one might expect from a relative cohomology theory; for instance, we get long exact sequences of relative cohomology groups in both variables in certain cases. Moreover, we get an Avramov–Martsinkovsky exact sequence for complexes, connecting the relative, the absolute and the Tate cohomological functors [6, 7.1]. Although our definition of relative Ext deals only with the special case when N is an R-module, it could possibly propose a new point of view for looking at relative Ext for complexes.

Using Vogel cohomology for modules, Martsinkovsky [17] introduced a generalization of the Auslander δ-invariant of a finitely generated R-module M over a Gorenstein local ring to arbitrary commutative noetherian local rings by taking into account the dimension of the k-vector spaces $\mathrm{Ker}\,(\tilde{\varepsilon}^n_R(M, k) : \mathrm{Ext}^n_R(M, k) \to \widetilde{\mathrm{Ext}}^n_R(M, k))$. He called them ξ-invariants. In Section 5, we use the same method to assign new homological invariants to any complex M of finite homology (i.e. $\mathrm{H}_i(M)$ finitely generated for all integers i) over a commutative noetherian local ring. This gives an extended version of the ξ-invariants in the setting of complexes. Furthermore, parallel to [6, Sec. 9], we introduce and study relative and Tate versions of Betti numbers for complexes. We study these invariants and compare them with the (absolute) Betti numbers of the complexes.

2. Preliminaries

In this paper R denotes an associative ring. The term R-module means a left R-module. R^{o} denotes the opposite ring. Note that a right R-module is a left R^{o}-module. We begin by recalling some basic definitions and results in the category of complexes of left R-modules. An R-complex X is a sequence of R-modules X_i and R-linear maps $\partial_i^X : X_i \to X_{i-1}$, $i \in \mathbb{Z}$. The module X_i is called the module in degree i and ∂_i^X is the ith differential. An R-module M is thought of as a complex concentrated in degree zero. For any integer m, $\Sigma^m X$ denotes the complex X shifted m degrees to the left, i.e. $(\Sigma^m X)_i = X_{i-m}$ and $\partial_i^{\Sigma^m X} = (-1)^m \partial^X_{i-m}$. We associate two numbers

$$\sup X = \sup\{i \in \mathbb{Z} \mid X_i \neq 0\} \quad \text{and} \quad \inf X = \inf\{i \in \mathbb{Z} \mid X_i \neq 0\}$$

with the complex X to compute its position. The complex X is called bounded above (resp. bounded below) if $\sup X < \infty$ (resp. $\inf X > -\infty$). It is bounded when it is bounded below and bounded above. For an R-complex X and $i \in \mathbb{Z}$ we set $C_i(X) = \mathrm{Coker}\,\partial^X_{i+1}$. The homology functor from R-complexes of R-modules to graded R-modules is as usual denoted by H( ). The homology complex $\mathrm{H}(X)$ is defined by setting $\mathrm{H}(X)_i = \mathrm{H}_i(X)$ and $\partial_i^{\mathrm{H}(X)} = 0$ for all $i \in \mathbb{Z}$. X is said to be homologically trivial if $\mathrm{H}(X) = 0$. X is called homologically bounded above (resp. bounded below, bounded) if the complex $\mathrm{H}(X)$ is so. X is called homologically finite if $\mathrm{H}_i(M)$ is finitely generated for all integers i.

A homomorphism $\varphi : X \to Y$ of degree i is a sequence of R-linear maps $\varphi_n : X_n \to Y_{n+i}$ for $n \in \mathbb{Z}$. All homomorphisms of degree i form an abelian group, denoted as $\mathrm{Hom}_R(X, Y)_i$, which we identify with $\prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(X_n, Y_{n+i})$. It appears as the ith component of a complex $\mathrm{Hom}_R(X, Y)$ of abelian groups, with differential $\partial(\varphi_n) = \partial^Y_{n+i}\varphi_n - (-1)^i\varphi_{n-1}\partial^X_n$ for $\varphi = (\varphi_n) \in \mathrm{Hom}_R(X, Y)_i$. A homomorphism $\varphi \in \mathrm{Hom}_R(X, Y)_n$ is called a *chain map* if $\partial(\varphi) = 0$. A *morphism* of complexes is a chain map of degree 0. The category of R-complexes and chain maps is denoted by $\mathcal{C}(R)$. A *quasi-isomorphism* $\varphi : X \to Y$ is a morphism such that $H(\varphi)$ is an isomorphism. Complexes X and Y are quasi-isomorphic (denoted as $X \simeq Y$) if they are linked by a chain of quasi-isomorphisms.

Following [5] we say that a complex P of R-modules is *semiprojective* if $\mathrm{Hom}_R(P,\ )$ preserves surjective quasi-isomorphisms. It is known [5, (8.5.1), (8.7.3)] that P is semiprojective if and only if each P_i is projective and $\mathrm{Hom}_R(P,\ )$ preserves quasi-isomorphisms.

A *semiprojective resolution* of complex M is a quasi-isomorphism of complexes $\pi : P \to M$, with P semiprojective; when π is surjective, the resolution is called strict. By [5, 8.3.3] we know that every complex M has a strict semiprojective resolution $P \to M$. If $\mathrm{H}(M)$ is bounded below, then P can be chosen so that $\inf P = \inf \mathrm{H}(M)$. If, in addition, R is left noetherian and $\mathrm{H}_i(M)$ is finitely generated for each $i \in \mathbb{Z}$, then P can be chosen so that each P_i is finitely generated. The projective dimension of M is defined by

$$\mathrm{pd}_R\, M = \inf\{\sup\{n \mid P_n \neq 0\} \mid P \text{ is a semiprojective resolution of } M\}.$$

Dually, a complex I of R-modules is called *semi-injective* if $\mathrm{Hom}_R(\ , I)$ transfers injective quasi-isomorphisms into surjective quasi-isomorphisms. It follows from [5, (9.5.1), (9.7.3)] that a complex I is semi-injective if and only if I_i is an injective R-module for each $i \in \mathbb{Z}$ and $\mathrm{Hom}_R(\ , I)$ preserves quasi-isomorphisms, which is equivalent to saying that I_i is injective for each $i \in \mathbb{Z}$ and $\mathrm{H}(\mathrm{Hom}_R(M, I)) = 0$ for every complex M with $\mathrm{H}(M) = 0$.

A *semi-injective resolution* of N is a quasi-isomorphism of complexes $i : N \to I$ with I semi-injective. When i is injective, the resolution is called strict. By [5, 9.3.3] every complex has a strict semi-injective resolution. When $\mathrm{H}(N)$ is bounded above, by [4, 1.7], I can be chosen so that $\sup I = \sup \mathrm{H}(N)$. The injective dimension of N is defined by

$$\mathrm{id}_R\, N = \inf\{\sup\{-n \mid I_n \neq 0\} \mid I \text{ is a semi-injective resolution of } N\}.$$

We conclude this preliminary section by recalling the notion of the right derived functors of the homomorphism functor, which is denoted by $\mathrm{Ext}^*_R(\ ,\ )$. Note that this can be computed using a semiprojective resolution of M as well as a semi-injective resolution of N; see [4, 1.8]. More precisely, given complexes M and N, the complex $\mathrm{Ext}^*_R(M, N)$ is defined uniquely (up to canonical isomorphism) by setting $\mathrm{Ext}^*_R(M, N) = \mathrm{H}^*(\mathrm{Hom}_R(P_M, N)) = \mathrm{H}^*(\mathrm{Hom}_R(M, I_N))$, where $P_M \to M$ is a semiprojective resolution of M and $N \to I_N$ is a semi-injective resolution of N.

3. Complete cohomology for complexes

Throughout the paper, as in [17], we shall use the symbol (A, B) for the graded Hom functor applied to the graded R-modules A and B. So, for all $i \in \mathbb{Z}$,

$$(A, B)_i = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(A_n, B_{n+i}).$$

3.1. Complete cohomology

Let M and N be two complexes and P_M and P_N denote their semiprojective resolutions, respectively. We shall use P_M (resp. P_N) to denote the corresponding underlying graded modules. The subset $(P_M, P_N)_{ba}$ of bounded above homogeneous maps (a homogeneous map is called bounded above if there exists an integer n such that $\alpha_i = 0$ for all $i > n$) is a graded submodule of (P_M, P_N). The restriction of ∂ to $(P_M, P_N)_{ba}$ makes it into a subcomplex of

(P_M, P_N). We denote by $(\widetilde{P_M, P_N})$ the quotient complex

$$(\widetilde{P_M, P_N}) = (P_M, P_N)/(P_M, P_N)_{ba}.$$

Passing on to cohomology we obtain complete (Vogel) cohomology, $\widetilde{\mathrm{Ext}}^*_R(M, N)$.

By essentially following the same argument as in the module case, one can see that $\widetilde{\mathrm{Ext}}^*_R$ is a cohomological functor, independent of the choice of semiprojective resolutions of M and N (see [13, Sec. I] for a proof of this fact in the module case).

It should be noted that when M and N are R-modules, our definition of Vogel Ext is compatible with the classical one, because in the module case, every map in (P_M, P_N) is naturally bounded below and so bounded above maps are precisely bounded ones.

3.1.1

There is already a notion of Vogel cohomology for complexes which is different from our definition [13, III]. The subcomplex that Goichot considered consists of bounded morphisms while we consider bounded above morphisms. With Goichot's construction, most of our results are not valid. Among them are Proposition 3.1.3, which indicates the rigidity of the complete cohomology and 3.1.7, which establish the equivalence between the complete cohomology and the Tate cohomology of [20, Sec. 4], when the complex in the first argument is of finite Gorenstein projective dimension.

3.1.2

The short exact sequence

$$0 \to (P_M, P_N)_{ba} \to (P_M, P_N) \to (\widetilde{P_M, P_N}) \to 0,$$

where the cohomology of the middle term is just $\mathrm{Ext}^*_R(M, N)$, yields, upon passing to the corresponding long cohomology exact sequence, a natural transformation

$$\tilde{\varepsilon}^*_R(M, N) : \mathrm{Ext}^*_R(M, N) \to \widetilde{\mathrm{Ext}}^*_R(M, N),$$

which, for any integer n, fits into the following long exact sequence

$$\mathrm{X}^n_R(M, N) \xrightarrow{\varepsilon^n_R(M,N)} \mathrm{Ext}^n_R(M, N) \xrightarrow{\tilde{\varepsilon}^n_R(M,N)} \widetilde{\mathrm{Ext}}^n_R(M, N) \longrightarrow \mathrm{X}^{n+1}_R(M, N),$$

where $\mathrm{X}^n_R(M, N)$ is the $(-n)$th cohomology group of the total complex $(P_M, P_N)_{ba}$.

Our first result, which establishes the rigidity of the complete cohomology for complexes, extends [15, 2.4] to complexes.

Proposition 3.1.3. *For any homologically bounded above complex M the following are equivalent.*

(i) $\mathrm{pd}_R\, M$ *is finite.*
(ii) $\widetilde{\mathrm{Ext}}^i_R(M, \ ) = 0$ *for all integers i.*
(iii) $\widetilde{\mathrm{Ext}}^i_R(\ , M) = 0$ *for all integers i.*
(iv) $\widetilde{\mathrm{Ext}}^0_R(M, M) = 0$.

Proof. (i) $\Rightarrow$ (ii). Since $\mathrm{pd}_R\, M < \infty$, there is a semiprojective resolution of M, say P_M, with $\sup P_M < \infty$. Hence $(P_M, \ )_{ba} = (P_M, \ )$. So $\widetilde{\mathrm{Ext}}^i_R(M, \ ) = 0$ for all $i \in \mathbb{Z}$.

(ii) $\Rightarrow$ (iv). This is clear.

(iv) $\Rightarrow$ (i). Since $\widetilde{\mathrm{Ext}}^0_R(M, M) = 0$, in view of 3.1.2, we may deduce that id^M, the identity map on M, is a cycle. So there should be a bounded above morphism ψ in $((P_M, P_M)_{ba})_0$ such that $\mathrm{id}^{P_M} - \psi \in \mathrm{Im}\, \partial_1$, where $\partial_1 : (P_M, P_M)_1 \to (P_M, P_M)_0$. Hence there exists $\varphi \in (P_M, P_M)_1$ such that $\partial_1 \varphi = \mathrm{id}^{P_M} - \psi$.

Since ψ is bounded above, for all $j \gg 0$, $(\mathrm{id}^{P_M} - \psi)_j$ is the identity morphism on P_j. So for all $j \gg 0$,

$$\varphi_{j-1}\partial^{P_M}_j + \partial^{P_M}_{j+1}\varphi_j = \mathrm{id}^{P_j}.$$

Since M is homologically bounded above, P_M is exact in all degrees large enough. Let j be an integer such that $\operatorname{Ker} \partial_j^{P_M} = \operatorname{Im} \partial_{j+1}^{P_M}$ and $\varphi_{j-1}\partial_j^{P_M} + \partial_{j+1}^{P_M}\varphi_j = \mathrm{id}^{P_j}$. Therefore, for all $x \in \operatorname{Im} \partial_{j+1}^{P_M}$, $\partial_{j+1}^{P_M}\varphi_j(x) = x$. Thus the map $P_{j+1} \to \operatorname{Im} \partial_{j+1}^{P_M}$ splits. Hence by [4, 2.4.P], $\operatorname{pd}_R M < \infty$.

The implications (i) ⇒ (iii) ⇒ (iv) ⇒ (i) can be proved similarly. □

3.1.4

It follows from the above proposition, in view of the long exact sequence of 3.1.2, that if $\operatorname{pd}_R M$ or $\operatorname{pd}_R N$ is finite, then for any integer n,

$$\mathrm{X}_R^n(M, N) \cong \operatorname{Ext}_R^n(M, N).$$

3.1.5

Like for the module case [13, Sec. I], one can see that there exist long exact sequences of complete cohomology groups associated with short exact sequences of complexes in either argument. Moreover, similar arguments can be applied to show that we have long exact sequences of X_R^* cohomology groups.

3.1.6

In [20, Sec. 4], Veliche introduced a Tate cohomology theory for complexes. Let us recall its construction, briefly. A totally acyclic complex T of projective modules is a complex of projective R-modules such that for any projective R-module Q and any integer $n \in \mathbb{Z}$, $\mathrm{H}_n(T) = \mathrm{H}_n(\operatorname{Hom}_R(T, Q)) = 0$. A *complete resolution* of M is a diagram of morphism of complexes

$$T \xrightarrow{\tau} P \xrightarrow{\pi} M$$

where $\pi : P \to M$ is a semiprojective resolution, T is a totally acyclic complex of projective modules and τ_i is bijective for all $i \gg 0$. A complete resolution is said to be surjective if τ_i is surjective for all $i \in \mathbb{Z}$. M is called of finite *Gorenstein projective dimension* (G-projective, for short) if it admits a complete resolution $T \xrightarrow{\tau} P \xrightarrow{\pi} M$. Let $\widetilde{\mathcal{GP}}(R)$ denote the class of complexes of finite Gorenstein projective dimension. Let M be a complex in $\widetilde{\mathcal{GP}}(R)$ and $T \xrightarrow{\tau} P \xrightarrow{\pi} M$ be a complete resolution of M. Let N be an arbitrary complex. For each $n \in \mathbb{Z}$, the nth Tate cohomology group is defined in [20, Sec. 4], by

$$\widehat{\operatorname{Ext}}_R^n(M, N) = \mathrm{H}_{-n}(\operatorname{Hom}_R(T, N)).$$

3.1.7

Using the same method as in [17, 2.1] one can see that for any complex M of finite Gorenstein projective dimension and any bounded below complex N our extension of Vogel cohomology for complexes is compatible with Veliche's extension of Tate cohomology; that is, for all integers n,

$$\widetilde{\operatorname{Ext}}_R^n(M, N) \cong \widehat{\operatorname{Ext}}_R^n(M, N).$$

3.2. Complete cohomology using injectives

Here we introduce a complete cohomology theory using injectives instead of projectives (for an analogue in the module case, see [19]). Let I_M and I_N be semi-injective resolutions of the complexes M and N, respectively. Let $(I_M, I_N)_{bb}$ denote the subcomplex of bounded below homomorphisms, which actually is the total complex. We shall use $\widetilde{(I_M, I_N)}$ to denote the quotient complex

$$\widetilde{(I_M, I_N)} = (I_M, I_N)/(I_M, I_N)_{bb}.$$

Now passing on to cohomology we obtain, for any $n \in \mathbb{Z}$, the cohomology group

$$\widetilde{\operatorname{ext}}_R^n(M, N) = \mathrm{H}_{-n}(\widetilde{(I_M, I_N)}).$$

One can verify that $\widetilde{\mathrm{ext}}^n_R(M, N)$ is independent of the choice of semi-injective resolutions of M and N. In fact, this is a cohomological functor contravariant in the first variable and covariant in the second variable.

The short exact sequence

$$0 \to (I_M, I_N)_{bb} \to (I_M, I_N) \to \widetilde{(I_M, I_N)} \to 0,$$

of complexes induces, for any integer $n \geq 0$, a long exact sequence of cohomological groups

$$\mathrm{x}^n_R(M, N) \xrightarrow{\gamma^n_R(M,N)} \mathrm{Ext}^n_R(M, N) \xrightarrow{\tilde{\gamma}^n_R(M,N)} \widetilde{\mathrm{ext}}^n_R(M, N) \longrightarrow \mathrm{x}^{n+1}_R(M, N),$$

where $\mathrm{x}^n_R(M, N)$ is the $(-n)$th cohomology group of the total complex $(I_M, I_N)_{bb}$.

Injective versions of 3.1.3 and 3.1.4 are valid here. In particular, any homologically bounded below complex N is of finite injective dimension if and only if $\widetilde{\mathrm{ext}}^0_R(N, N) = 0$. This, in the case of modules, specializes to [19, 3.7]. We also note that one can get long exact sequences of both cohomological functors $\widetilde{\mathrm{ext}}^*_R$ and x^*_R in either variable.

3.2.1

Like Veliche's extension of Tate cohomology, one can consider complete coresolutions of complexes of finite Gorenstein injective dimension and define Tate cohomology functors $\widehat{\mathrm{ext}}^n_R$; see [1, 2.2] for details. Since we need this construction, we review it here. A totally acyclic complex of injectives is a complex T of injective R-modules such that for any injective R-module I and any $n \in \mathbb{Z}$, $\mathrm{H}_n(T) = \mathrm{H}_n(\mathrm{Hom}_R(I, T)) = 0$. Let N be an R-complex. A *complete coresolution* of N is a diagram of complexes $N \xrightarrow{l} I \xrightarrow{\nu} T$ where $N \xrightarrow{l} I$ is a semi-injective resolution of N, T is a totally acyclic complex of injective R-modules and ν_i is bijective for all $i \ll 0$. An injective complete coresolution is a complete coresolution such that ν_i is injective for all $i \in \mathbb{Z}$. We always can get an injective complete coresolution $N \xrightarrow{l} I \xrightarrow{\nu'} T'$ from any complete coresolution $N \xrightarrow{l} I \xrightarrow{\nu} T$ in such a way that T and T' are homology equivalent. So we may always assume that $N \xrightarrow{l} I \xrightarrow{\nu} T$ is an injective complete coresolution. N is called of finite *Gorenstein injective dimension* (G-injective dimension, for short) if it admits a complete coresolution. Set $L = \mathrm{Coker}\,\nu$ to get a short exact sequence of complexes

$$0 \to I \xrightarrow{\nu} T \to L \to 0.$$

Since ν is split in each degree, L is a complex of injective modules.

Let N be a complex of finite Gorenstein injective dimension. Choose a complete coresolution $N \xrightarrow{l} I \xrightarrow{\nu} T$. For each $n \in \mathbb{Z}$ and any complex M a Tate cohomology group is defined in [1] by the equality

$$\widehat{\mathrm{ext}}^n_R(M, N) = \mathrm{H}_{-n}(\mathrm{Hom}_R(M, T)).$$

3.2.2

Like for 3.1.7, it can be seen that for any complex N of finite Gorenstein injective dimension and any bounded above complex M,

$$\widetilde{\mathrm{ext}}^n_R(M, N) \cong \widehat{\mathrm{ext}}^n_R(M, N),$$

for all integers n.

3.3. Comparisons

Our aim here is to compare cohomological functors $\widetilde{\mathrm{Ext}}^i_R$ and $\widetilde{\mathrm{ext}}^i_R$, as well as X^i_R and x^i_R. We begin by recalling two invariants associated with an associative ring R, i.e. $\mathrm{spli}(R)$, the supremum of the projective lengths of injective modules, and $\mathrm{silp}(R)$, the supremum of the injective lengths of projective modules. It is known that if these two numbers are finite, then they are equal [12, 1.6].

The next two results are proved in [1, 3.2 and 3.3] for when R is noetherian (left and right). Their proofs work without any assumption on the ring. Since we need them in this generality, we restate them here.

Proposition 3.3.1. *Let R be a ring. Then the following are equivalent.*

(i) $\mathrm{spli}(R) = \mathrm{silp}(R) < \infty$.
(ii) *Every homologically bounded above complex M has finite G-projective dimension.*
(iii) *Every homologically bounded below complex N has finite G-injective dimension.*

Theorem 3.3.2. *Let R be a ring with the property that* $\mathrm{spli}(R) = \mathrm{silp}(R) < \infty$. *Let M be a bounded above complex and N be a bounded below complex. Then for each $n \in \mathbb{Z}$,*

$$\widehat{\mathrm{Ext}}^n_R(M, N) \cong \widehat{\mathrm{ext}}^n_R(M, N).$$

The next result is proved for modules in [19, 5.2].

Theorem 3.3.3. *Let R be any ring. Then* $\mathrm{spli}(R) = \mathrm{silp}(R) < \infty$ *if and only if for each bounded above complex M and each bounded below complex N,*

$$\widetilde{\mathrm{Ext}}^n_R(M, N) \cong \widetilde{\mathrm{ext}}^n_R(M, N),$$

for all integers n.

Proof. First assume that $\mathrm{spli}(R) = \mathrm{silp}(R) < \infty$. So by Proposition 3.3.1, every homologically bounded above (below) complex M (N) has finite G-projective (G-injective) dimension. Hence by 3.1.7 and 3.2.2, we have isomorphisms

$$\widetilde{\mathrm{Ext}}^n_R(M, N) \cong \widehat{\mathrm{Ext}}^n_R(M, N) \quad \text{and} \quad \widetilde{\mathrm{ext}}^n_R(M, N) \cong \widehat{\mathrm{ext}}^n_R(M, N).$$

Hence Theorem 3.3.2 implies the result. The converse follows from [19, 5.2]. $\square$

The proof of the following proposition is based on the same technique as in the proof of [17, 2.1].

Proposition 3.3.4. *Let M be a complex of finite Gorenstein projective dimension. Then for each homologically bounded below complex N and each integer n, there exist natural in M and N isomorphisms*

$$\mathrm{X}^{n+1}_R(M, N) \cong \mathrm{H}_{-n}\mathrm{Hom}_R(L, P_N),$$

where $T \xrightarrow{\tau} P_M \xrightarrow{\pi} M$ is a surjective complete resolution of M, $L = \mathrm{Ker}\,\tau$ and P_N is a semiprojective resolution of N. In the case where N itself is bounded below, we have

$$\mathrm{X}^{n+1}_R(M, N) \cong \mathrm{H}_{-n}\mathrm{Hom}_R(L, N).$$

Proof. Consider the split exact sequence $0 \to L \to T \to P_M \to 0$ of complexes. Let P_N denote the semiprojective resolution of N and apply the Hom functor $(\ , P_N)$ on this exact sequence to get the following commutative diagram of Hom groups:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (P_M, P_N)_{ba} \arrow[r] \arrow[d] & (P_M, P_N) \arrow[r] \arrow[d] & \widetilde{(P_M, P_N)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (T, P_N)_{ba} \arrow[r] \arrow[d] & (T, P_N) \arrow[r] \arrow[d] & \widetilde{(T, P_N)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (L, P_N)_{ba} \arrow[r] \arrow[d] & (L, P_N) \arrow[r] \arrow[d] & \widetilde{(L, P_N)} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which all rows and columns are exact. Since N is bounded below, the morphisms in (T, P_N) are bounded below. So bounded above ones are in fact bounded (i.e. only finitely many components of them are different from zero). So the complex $(T, P_N)_{ba}$ is isomorphic to $(T, R) \otimes_R P_N$. The first factor is exact, since T is a totally acyclic complex. The second factor also is exact because P_N is a bounded below complex of projective modules. Hence the complex $(T, P_N)_{ba}$ is exact. Therefore the exactness of the first column implies that for any integer n, the $(-n-1)$th cohomology of the complex $(P_M, P_N)_{ba}$, which is $\mathrm{X}_R^{n+1}(M, N)$, coincides with the $(-n)$th cohomology of the complex $(L, P_N)_{ba}$. Since L is a bounded above complex, every morphism in (L, P_N) is bounded above and so $\widetilde{(L, P_N)} = 0$. So the exactness of the last row implies that for any integer n, the cohomologies of the complexes $(L, P_N)_{ba}$ and (L, P_N) coincide. Hence the result follows. For the proof of our last claim note that L is a complex of projectives and so $\mathrm{Hom}_R(L, \ )$ preserves quasi-isomorphisms between bounded below complexes. That is $\mathrm{H}_{-n}\mathrm{Hom}_R(L, P_N) \cong \mathrm{H}_{-n}\mathrm{Hom}_R(L, N)$. □

In the case when R is noetherian and M is finite, the following corollary specializes to [6, 7.1].

Corollary 3.3.5. *Let M be a complex of finite Gorenstein projective dimension. Then for each homologically bounded below complex N and for each $n > \mathrm{Gpd}_R\ M - \inf \mathrm{H}(N)$, $\mathrm{X}_R^n(M, N) = 0$. In particular, for such n's, $\mathrm{Ext}_R^n(M, N) \cong \widetilde{\mathrm{Ext}}_R^n(M, N)$.*

Proof. By the above proposition, $\mathrm{X}_R^n(M, N) \cong \mathrm{H}_{-n+1}\mathrm{Hom}_R(L, P_N)$, where $T_M \xrightarrow{\tau} P_M \to M$ is a surjective complete resolution of M, $L = \mathrm{Ker}\,\tau$ and P_N is a semiprojective resolution of N. Since $\sup L = \mathrm{Gpd}_R\ M - 1$, and $\inf P_N = \inf \mathrm{H}(N)$, we have $\inf \mathrm{Hom}_R(L, P_N) \geq \inf \mathrm{H}(N) - \sup L$. So for $n < \inf \mathrm{H}(N) - \mathrm{Gpd}_R\ M$, $\mathrm{Hom}_R(L, P_N)_n = 0$. The result now follows from the above proposition. Our last claim follows from the long exact sequence of 3.1.2. □

3.3.6

In view of 3.1.4, [20, 3.8] and the above corollary, for any complex M of finite Gorenstein projective dimension, we have

$$\mathrm{Gpd}_R\ M = \sup\{n + \inf \mathrm{H}(N) \mid \mathrm{X}_R^n(M, N) \neq 0 \text{ for some homologically bounded below complex } N\}.$$

We also have the following dual result. Its proof parallels that of 3.3.4 and so is omitted.

Proposition 3.3.7. *Let N be a complex of finite Gorenstein injective dimension. Then for each homologically bounded above complex M and each integer n, there are natural in M and N isomorphisms*

$$\mathrm{x}_R^{n+1}(M, N) \cong \mathrm{H}_{-n}\mathrm{Hom}_R(I_M, K),$$

where $N \xrightarrow{\iota} I_N \xrightarrow{\upsilon} T$ is a surjective complete coresolution of N, $K = \mathrm{Coker}\,\upsilon$ and I_M is a semi-injective resolution of M. In the case where M is bounded above, we have

$$\mathrm{x}_R^{n+1}(M, N) \cong \mathrm{H}_{-n}\mathrm{Hom}_R(M, K).$$

In view of this proposition, we are able to record the following corollary.

Corollary 3.3.8. *Let N be a complex of finite Gorenstein injective dimension. Then for each homologically bounded above complex M and each $n > \sup \mathrm{H}(M) + \mathrm{Gid}_R\ N$, $\mathrm{x}_R^n(M, N) = 0$. In particular, for such n's, $\mathrm{Ext}_R^n(M, N) \cong \widetilde{\mathrm{ext}}_R^n(M, N)$.*

Hence, by [1, 2.4] and the above corollary, for any complex N of finite Gorenstein injective dimension, we have

$$\mathrm{Gid}_R\ N = \sup\{n - \sup \mathrm{H}(M) \mid \mathrm{x}_R^n(M, N) \neq 0 \text{ for some homologically bounded above complex } M\}.$$

Theorem 3.3.9. *Let M and N be complexes such that $\mathrm{Gpd}_R\ M < \infty$ and $\mathrm{Gid}_R\ N < \infty$. Then for any integer n,*

$$\mathrm{X}_R^n(M, N) \cong \mathrm{x}_R^n(M, N).$$

Proof. Let $T_M \xrightarrow{\tau} P_M \to M$ be a surjective complete resolution of M, $L = \operatorname{Ker}\tau$, and P_N be a semiprojective resolution of N. Set $\operatorname{Gpd}_R M = g$ and $\inf P_N = \inf \mathrm{H}(N) = t$. By 3.3.4, for any integer $n \in \mathbb{Z}$, $\mathrm{X}_R^{n+1}(M, N) \cong \mathrm{H}_{-n}\mathrm{Hom}_R(L, P_N)$. But, in order to compute $\mathrm{H}_{-n}\mathrm{Hom}_R(L, P_N)$, one just needs to consider $L' = L_{n+t-1} \sqsupset$, the hard right truncation of L at $n + t - 1$. Since L' is a bounded complex of projectives, it is semiprojective. Hence we can replace P_N by a semi-injective resolution of N, say I_N. So $\mathrm{H}_{-n}\mathrm{Hom}_R(L, P_N) \cong \mathrm{H}_{-n}\mathrm{Hom}_R(L', I_N)$. Let $N \to I_N \xrightarrow{\upsilon} T'_N$ be an injective complete coresolution of N, set $K = \operatorname{Coker}\upsilon$ and consider the exact sequence of complexes of injective modules

$$0 \to I_N \to T'_N \to K \to 0.$$

Since T'_N is an exact complex, for any integer i, $\mathrm{H}_i\mathrm{Hom}_R(L', T_N) = 0$. So it follows from the above short exact sequence of complexes that

$$\mathrm{H}_{-n}\mathrm{Hom}_R(L', I_N) \cong \mathrm{H}_{-n+1}\mathrm{Hom}_R(L', K).$$

Now look at $\mathrm{x}_R^n(M, N)$. By Proposition 3.3.7, $\mathrm{x}_R^{n+1}(M, N) \cong \mathrm{H}_{-n}\mathrm{Hom}_R(I_M, K)$, where I_M is a semi-injective resolution of M. Set $\sup I_M = s$. So to compute $\mathrm{H}_{-n}\mathrm{Hom}_R(I_M, K)$, we just need to consider $K' = \sqsubset_{s-n+1} K$, which is the hard left truncation of K at $s - n + 1$. In fact

$$\mathrm{H}_{-n}\mathrm{Hom}_R(I_M, K) = \mathrm{H}_{-n}\mathrm{Hom}_R(I_M, K').$$

Note that for any complex C with $\sup C \le \sup I_M = s$, one can replace K by K'. Since K' is a bounded complex of injectives, it is semi-injective. Therefore we can (and do) replace I_M by semiprojective resolution P_M of M, so $\mathrm{H}_{-n}\mathrm{Hom}_R(I_M, K') \cong \mathrm{H}_{-n}\mathrm{Hom}_R(P_M, K')$. Since T_M is exact, the short exact sequence $0 \to L \to T_M \to P_M \to 0$ implies that

$$\mathrm{H}_{-n}\mathrm{Hom}_R(P_M, K') \cong \mathrm{H}_{-n+1}\mathrm{Hom}_R(L, K').$$

On the other hand, since $\inf K' = -l + 1 \ge \inf P_N = t$ and $\sup L' = g - 1 \le \sup I_M = s$, as above we may deduce that

$$\mathrm{H}_{-n+1}\mathrm{Hom}_R(L, K') \cong \mathrm{H}_{-n+1}\mathrm{Hom}_R(L', K') \quad \text{and} \quad \mathrm{H}_{-n+1}\mathrm{Hom}_R(L', K) \cong \mathrm{H}_{-n+1}\mathrm{Hom}_R(L', K').$$

The proof is hence complete. □

Corollary 3.3.10. *Let M and N be complexes such that $\operatorname{Gpd}_R M < \infty$ and $\operatorname{Gid}_R N < \infty$. Then for all integers $n > \min\{\operatorname{Gpd}_R M - \inf \mathrm{H}(N), \operatorname{Gid}_R N + \sup \mathrm{H}(M)\}$,*

$$\widetilde{\mathrm{Ext}}_R^n(M, N) \cong \mathrm{Ext}_R^n(M, N) \cong \widetilde{\mathrm{ext}}_R^n(M, N).$$

Proof. The result follows from 3.3.5 and 3.3.8, in conjunction with the balance of the above theorem. □

4. Relative cohomology for complexes

In this section, we introduce a notion of relative Ext functors for complexes. We keep the notation of previous sections. In particular, by 3.1.2, for all complexes M and N and all integers n, there exists an exact sequence of cohomological functors

$$\mathrm{X}_R^n(M, N) \xrightarrow{\varepsilon_R^n(M,N)} \mathrm{Ext}_R^n(M, N) \xrightarrow{\tilde{\varepsilon}_R^n(M,N)} \widetilde{\mathrm{Ext}}_R^n(M, N) \longrightarrow \mathrm{X}_R^{n+1}(M, N).$$

Definition 4.1. Let M be a complex with $\operatorname{Gpd}_R M = g < \infty$. For each $n \in \mathbb{Z}$ and each R-module N, we define the nth relative cohomology group $\mathrm{Ext}_{\mathcal{GP}}^n(M, N)$ as follows:

- If $\inf \mathrm{H}(M) = -\infty$, we set $\mathrm{Ext}_{\mathcal{GP}}^n(M, N) := \mathrm{X}_R^n(M, N)$.
- If $\inf \mathrm{H}(M) = t > -\infty$, we define it as follows:

$$\mathrm{Ext}_{\mathcal{GP}}^n(M, N) = \begin{cases} \mathrm{X}_R^n(M, N) & \text{for } n > t + 1; \\ \operatorname{Ker} \tilde{\varepsilon}_R^{t+1}(M, N) & \text{for } n = t + 1; \\ \mathrm{Ext}_R^n(M, N) & \text{for } n \le t. \end{cases}$$

If M and M' are two complexes such that $\inf \mathrm{H}(M) = \inf \mathrm{H}(M')$ and $\mu : M' \to M$ is a morphism of complexes, then for any integer n and any R-module homomorphism $\psi : N \to N'$, the homomorphism

$$\mathrm{Ext}^n_{\mathcal{GP}}(\mu, \psi) : \mathrm{Ext}^n_{\mathcal{GP}}(M, N) \to \mathrm{Ext}^n_{\mathcal{GP}}(M', N')$$

of cohomology groups is defined as follows:

- If $\inf \mathrm{H}(M) = \inf \mathrm{H}(M') = -\infty$, $\mathrm{Ext}^n_{\mathcal{GP}}(\mu, \psi) = \mathrm{X}^n_R(\mu, \psi)$.
- If $\inf \mathrm{H}(M) = \inf \mathrm{H}(M') = t > -\infty$,

$$\mathrm{Ext}^n_{\mathcal{GP}}(\mu, \psi) = \begin{cases} \mathrm{X}^n_R(\mu, \psi) & \text{for } n > t+1; \\ \mathrm{Ker}\, \tilde{\varepsilon}^{t+1}_R(\mu, \psi) & \text{for } n = t+1; \\ \mathrm{Ext}^n_R(\mu, \psi) & \text{for } n \le t. \end{cases}$$

Moreover, for any integer n, we define a comparison morphism

$$\varepsilon^n_{\mathcal{GP}}(M, N) : \mathrm{Ext}^n_{\mathcal{GP}}(M, N) \to \mathrm{Ext}^n_R(M, N)$$

by the following setting:

- If $\inf \mathrm{H}(M) = -\infty$, we set $\varepsilon^n_{\mathcal{GP}}(M, N) = \varepsilon^n_R(M, N)$.
- If $\inf \mathrm{H}(M) = t > -\infty$, we set

$$\varepsilon^n_{\mathcal{GP}}(M, N) = \begin{cases} \varepsilon^n_R(M, N) & \text{for } n > t+1; \\ i_{t+1} & \text{for } n = t+1; \\ \mathrm{id}^{\mathrm{Ext}^n_R(M,N)} & \text{for } n \le t \end{cases}$$

where $i_{t+1} : \mathrm{Ext}^{t+1}_{\mathcal{GP}}(M, N) \to \mathrm{Ext}^{t+1}_R(M, N)$ is the canonical inclusion.

Notation 4.2. *For any integer $t \in \mathbb{Z} \cup \{-\infty\}$, we let $\widetilde{\mathcal{GP}}(R)_t$ denote the full subcategory of $\mathcal{C}(R)$ whose objects are complexes M of finite Gorenstein projective dimension with* $\inf \mathrm{H}(M) = t$.

Using the above notation and conventions, we have the following theorem. The symbol $\mathcal{M}(R)$ denotes the category of R-modules and R-homomorphisms.

Theorem 4.3. *For each $n \in \mathbb{Z}$, the assignment $(M, N) \longmapsto \mathrm{Ext}^n_{\mathcal{GP}}(M, N)$ defines a functor*

$$\mathrm{Ext}^n_{\mathcal{GP}} : \widetilde{\mathcal{GP}}(R)^{\mathrm{op}}_t \times \mathcal{M}(R) \to \mathcal{M}(\mathbb{Z})$$

and the maps $\varepsilon^n_{\mathcal{GP}}(M, N)$ yield a morphism of functors $\varepsilon^n_{\mathcal{GP}} : \mathrm{Ext}^n_{\mathcal{GP}} \to \mathrm{Ext}^n_R$.

The module version of the following construction can be found in [6, 3.8].

Construction 4.4. Let M be a homologically bounded below complex of finite Gorenstein projective dimension. Let $\inf \mathrm{H}(M) = t > -\infty$ and $\mathrm{Gpd}_R\, M = g < \infty$. It is known that in this case there exists a surjective complete resolution $T \xrightarrow{\tau} P \xrightarrow{\pi} M$ with $\inf P = t$ and τ_n bijective for all $n \ge g$. Set $L = \mathrm{Ker}\, \tau$. Let χ denote the inclusion $L \subseteq T$, and set

$$G_n = \begin{cases} L_{n-1} & \text{for } n > t; \\ C_t(T) & \text{for } n = t; \\ 0 & \text{for } n < t \end{cases} \quad \text{and} \quad \partial^G_n = \begin{cases} -\partial^L_{n-1} & \text{for } n > t+1; \\ -(\Omega^t \chi) \circ \omega^L_t & \text{for } n = t+1; \\ 0 & \text{for } n < t+1 \end{cases}$$

where $\omega^L_t : L_t \to C_t(L)$ is the canonical map of modules. Also set

$$T^\flat_n = \begin{cases} T_n & \text{for } n \ge t; \\ C_t(T) & \text{for } n = t-1; \\ 0 & \text{for } n \le t-2 \end{cases} \quad \text{and} \quad \partial^{T^\flat}_n = \begin{cases} \partial^T_n & \text{for } n > t; \\ \omega^T_t & \text{for } n = t; \\ 0 & \text{for } n < t \end{cases}$$

where $\omega^T_t : T_t \to C_t(T)$ is the canonical map of modules. Finally set

$$\chi^\flat_n = \begin{cases} \chi_n & \text{for } n \ge t; \\ id^{C_t(T)} & \text{for } n = t-1; \\ 0 & \text{for } n \le t-2 \end{cases} \quad \text{and} \quad \tau^\flat_n = \begin{cases} \tau_n & \text{for } n \ge t; \\ \Omega^t \tau & \text{for } n = t-1; \\ 0 & \text{for } n \le t-2. \end{cases}$$

It is easy to see that $\chi^\flat : \Sigma^{-1}G \to T^\flat$ and $\tau^\flat : T^\flat \to P$ are morphisms of complexes and the sequence $0 \to \Sigma^{-1}G \xrightarrow{\chi^\flat} T^\flat \xrightarrow{\tau^\flat} P \to 0$ is split exact.

4.5

When M is a homologically bounded below complex of finite Gorenstein projective dimension it is tempting, as in the module case, to use $\mathrm{Hom}_R(G, N)$ to define relative cohomology groups for complexes, where G is introduced in the above construction. Now assume that M' is another complex with $\mathrm{Gpd}_R\, M' < \infty$ and $\inf \mathrm{H}(M) = \inf \mathrm{H}(M') > -\infty$. By the above construction there exists a split short exact sequence $0 \to \Sigma^{-1}G' \xrightarrow{\chi^\flat} T'^\flat \xrightarrow{\tau^\flat} P' \to 0$ related to M'. Let $\mu : M' \to M$ be a morphism of complexes. Then, it is easy to see that there exist induced morphisms $\bar{\mu}$, $\hat{\mu}$ and $\check{\mu}$ such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{-1}G' \arrow[r] \arrow[d, "\check{\mu}"] & T'^\flat \arrow[r] \arrow[d, "\hat{\mu}"] & P' \arrow[r] \arrow[d, "\bar{\mu}"] & 0 \\
0 \arrow[r] & \Sigma^{-1}G \arrow[r] & T^\flat \arrow[r] & P \arrow[r] & 0.
\end{tikzcd} \qquad (*)$$

It is known that $\bar{\mu}$ and $\hat{\mu}$ are unique up to homotopy. But, as is also mentioned in [20, Sec. 6], we do not know whether such uniqueness holds for $\check{\mu}$, although we know that if μ is an isomorphism (in particular, if $\mu = id^M$), then so is $\check{\mu}^* = \mathrm{H}(\mathrm{Hom}_R(\check{\mu}, N))$. To overcome this hurdle, we used the structure of Vogel cohomology in our definition of relative cohomology functors.

Theorem 4.6. *With the notation of 4.4 and 4.5, for any complex $M \in \widetilde{\mathcal{GP}}(R)_t$ and any R-module N, there exists an isomorphism*

$$\rho(M, N) : \mathrm{H}_{-t}\mathrm{Hom}_R(\Sigma^{-1}G, N) \to \mathrm{Ext}^{t+1}_{\mathcal{GP}}(M, N)$$

such that for any complex $M' \in \widetilde{\mathcal{GP}}(R)_t$ and any morphism $\mu : M' \to M$, the diagram

$$\begin{tikzcd}
\mathrm{H}_{-t}\,\mathrm{Hom}_R(\Sigma^{-1}G, N) \arrow[r, "\mathrm{H}_{-t}\,\mathrm{Hom}_R(\check{\mu}, N)"] \arrow[d, "\rho(M,N)"'] & \mathrm{H}_{-t}\,\mathrm{Hom}_R(\Sigma^{-1}G', N) \arrow[d, "\rho(M',N)"'] \\
\mathrm{Ext}^{t+1}_{\mathcal{GP}}(M, N) \arrow[r, "\mathrm{Ext}^{t+1}_{\mathcal{GP}}(\mu, id^N)"] & \mathrm{Ext}^{t+1}_{\mathcal{GP}}(M', N)
\end{tikzcd}$$

is commutative.

Proof. Apply the left exact functor $\mathrm{Hom}_R(\ , N)$ to the diagram $(*)$ above, to get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(P, N) \arrow[r] \arrow[d, "\mathrm{Hom}_R(\bar{\mu},N)"'] & \mathrm{Hom}_R(T^\flat, N) \arrow[r] \arrow[d, "\mathrm{Hom}_R(\hat{\mu},N)"'] & \mathrm{Hom}_R(\Sigma^{-1}G, N) \arrow[r] \arrow[d, "\mathrm{Hom}_R(\check{\mu},N)"'] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(P', N) \arrow[r] & \mathrm{Hom}_R(T'^\flat, N) \arrow[r] & \mathrm{Hom}_R(\Sigma^{-1}G', N) \arrow[r] & 0
\end{tikzcd}$$

of complexes of abelian groups. This, in turn, induces the following commutative diagram of cohomology groups:

$$\begin{tikzcd}
\mathrm{H}_{-t}(\mathrm{Hom}_R(T^\flat, N)) \arrow[r] \arrow[d, "\hat{\mu}^*_{-t}"'] & \mathrm{H}_{-t}(\mathrm{Hom}_R(\Sigma^{-1}G, N)) \arrow[r] \arrow[d, "\mathrm{H}_{-t}\,\mathrm{Hom}_R(\check{\mu},N)"'] & \mathrm{H}_{-t-1}(\mathrm{Hom}_R(P, N)) \arrow[r] \arrow[d, "\bar{\mu}^*_{-t-1}"'] & \mathrm{H}_{-t-1}(\mathrm{Hom}_R(T^\flat, N)) \arrow[d, "\hat{\mu}^*_{-t-1}"'] \\
\mathrm{H}_{-t}(\mathrm{Hom}_R(T'^\flat, N)) \arrow[r] & \mathrm{H}_{-t}(\mathrm{Hom}_R(\Sigma^{-1}G', N)) \arrow[r] & \mathrm{H}_{-t-1}(\mathrm{Hom}_R(P', N)) \arrow[r] & \mathrm{H}_{-t-1}(\mathrm{Hom}_R(T'^\flat, N)).
\end{tikzcd}$$

By identifying the groups and maps appearing in the above diagram we get

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}_{-t}(\mathrm{Hom}_R(\Sigma^{-1}G, N)) \arrow[r, "\alpha"] \arrow[d, "\mathrm{H}_{-t}\,\mathrm{Hom}_R(\check{\mu},N)"'] & \mathrm{Ext}^{t+1}_R(M, N) \arrow[r, "\hat{\varepsilon}^{t+1}(\mu,N)"] \arrow[d, "\bar{\mu}^*_{-t-1}"'] & \widehat{\mathrm{Ext}}^{t+1}_R(M, N) \arrow[d, "\hat{\mu}^*_{-t-1}"'] \\
0 \arrow[r] & \mathrm{H}_{-t}(\mathrm{Hom}_R(\Sigma^{-1}G', N)) \arrow[r] & \mathrm{Ext}^{t+1}_R(M', N) \arrow[r] & \widehat{\mathrm{Ext}}^{t+1}_R(M', N).
\end{tikzcd}$$

So if we define $\rho(M, N) : \mathrm{H}_{-t}(\mathrm{Hom}_R(\Sigma^{-1}G, N)) \to \mathrm{Ext}^{t+1}_{\mathcal{GP}}(M, N)$, by setting $\rho(M, N)(x) = \alpha(x)$, the result follows from the exactness of the upper row of the above diagram and 3.1.7. □

Theorem 4.7. *Let M be a homologically bounded below complex of finite Gorenstein projective dimension. With the notation of Construction 4.4, for any integer n and any R-module N, we have*

$$\mathrm{Ext}^n_{\mathcal{GP}}(M, N) \cong \mathrm{H}_{-n}\mathrm{Hom}_R(G, N).$$

Proof. The result is clear for $n < t$. Assume that $n = t$. It follows from the left exactness of the Hom functor that $\mathrm{H}_{-t}\mathrm{Hom}_R(T^\flat, N) = 0$. So in view of the exact sequence $0 \to \Sigma^{-1}G \to T^\flat \to P \to 0$, we have

$$\mathrm{H}_{-t+1}\mathrm{Hom}_R(\Sigma^{-1}G, N) \cong \mathrm{H}_{-t}\mathrm{Hom}_R(P, N).$$

The right hand side is $\mathrm{Ext}^t_R(M, N)$ while for the left hand side we have

$$\mathrm{H}_{-t+1}\mathrm{Hom}_R(\Sigma^{-1}G, N) = \mathrm{H}_{-t+1}\Sigma^{-1}\mathrm{Hom}_R(G, N) = \mathrm{H}_{-t}\mathrm{Hom}_R(G, N).$$

The result hence follows in this case. Case $n = t + 1$ follows from Theorem 4.6. So assume that $n > t + 1$. By definition we have $\mathrm{Ext}^n_{\mathcal{GP}}(M, N) = \mathrm{X}^n_R(M, N)$. By Proposition 3.3.4, $\mathrm{X}^n_R(M, N) \cong \mathrm{H}_{-n+1}\mathrm{Hom}_R(L, N)$. But since $n > t + 1$, as we saw in the proof of Theorem 3.3.9, we have $\mathrm{H}_{-n+1}\mathrm{Hom}_R(L, N) = \mathrm{H}_{-n+1}\mathrm{Hom}_R(L', N) = \mathrm{H}_{-n+1}\mathrm{Hom}_R(\Sigma^{-1}G, N)$, where L' denotes the hard right truncation $L_t \sqsupset$ of L. The right hand side is equal to $\mathrm{H}_{-n}\mathrm{Hom}_R(G, N)$. The proof is hence complete. □

It follows, from the Definition 4.1 and the above theorem, that our definition of relative Ext is equivalent to that of [20, Sec. 6], when M is an R-module.

The next proposition extends to complexes a result of Avramov and Martsinkovsky [6, 4.2(3)].

Proposition 4.8. *Let M be a complex of finite projective dimension. Then for each integer n, and each R-module N,*

$$\mathrm{Ext}^n_{\mathcal{GP}}(M, N) \cong \mathrm{Ext}^n_R(M, N).$$

Proof. Since $\mathrm{pd}_R\, M < \infty$, by Proposition 3.1.3, $\widetilde{\mathrm{Ext}}^n_R(M, \ ) = 0$ for all integers n. If $\inf \mathrm{H}(M) = -\infty$, the result follows from the exact sequence of cohomology groups of 3.1.2. So assume that $\inf \mathrm{H}(M) = t > -\infty$. In this case, for $n \le t$, the isomorphism is clear. The case $n = t + 1$ also follows from the definition because $\mathrm{Ext}^{t+1}_{\mathcal{GP}}(M, N) = \mathrm{Ker}\,(\mathrm{Ext}^{t+1}_R(M, N) \to \widetilde{\mathrm{Ext}}^{t+1}_R(M, N))$ and $\widetilde{\mathrm{Ext}}^{t+1}_R(M, N) = 0$. The result for $n > t + 1$ follows from the long exact sequence of 3.1.2. □

The first statement of the following result extends [6, 4.2(4)] to the setting of complexes. Parts (i) and (ii) of it can be proved as in the module case, while part (iii) follows from [20, 3.8] in view of (i) and (ii).

Proposition 4.9. *Let M be a complex of finite Gorenstein projective dimension and N be an R-module.*

(i) *If $\mathrm{pd}_R\, N < \infty$, $\mathrm{Ext}^n_{\mathcal{GP}}(M, N) \cong \mathrm{Ext}^n_R(M, N)$ for all integers $n \in \mathbb{Z}$.*
(ii) *$\mathrm{Ext}^n_{\mathcal{GP}}(M, N) = 0$ for all integers $n > \mathrm{Gpd}_R\, M$.*
(iii) *$\mathrm{Gpd}_R\, M = \sup\{n \in \mathbb{Z} \mid \mathrm{Ext}^n_{\mathcal{GP}}(M, X) \neq 0$ for some R-module $X\}$.*

Exact sequences of relative groups exist in certain cases. Recall that a complex C is called $\mathcal{GP}$-proper exact if the induced complex $\mathrm{Hom}_R(G, C)$ is exact for all Gorenstein projective modules G.

Proposition 4.10. *Let M be a complex of finite Gorenstein projective dimension. Then for each $\mathcal{GP}$-proper exact sequence $\mathbf{N} = 0 \to N \to N' \to N'' \to 0$ of R-modules there exist natural in M and $\mathbf{N}$ homomorphisms $\partial^n_{\mathcal{GP}}(M, \mathbf{N})$ such that the sequence below is exact:*

$$\cdots \longrightarrow \mathrm{Ext}^n_{\mathcal{GP}}(M, N) \longrightarrow \mathrm{Ext}^n_{\mathcal{GP}}(M, N') \longrightarrow \mathrm{Ext}^n_{\mathcal{GP}}(M, N'')$$
$$\xrightarrow{\partial^n_{\mathcal{GP}}(M,\mathbf{N})} \mathrm{Ext}^{n+1}_{\mathcal{GP}}(M, N) \longrightarrow \cdots .$$

Proof. The result is clear from the long exact sequence of X_R, when $\inf \mathrm{H}(M) = -\infty$. So assume that $\inf \mathrm{H}(M) = t > -\infty$. The result trivially holds when $n < t$ and follows from the long exact sequence of X_R, when $n > t+1$. So we should consider the cases $n = t$ and $n = t+1$. Applying the functor $\mathrm{Hom}_R(\Sigma^{-1}G, \ )$ to the proper exact sequence $\mathbf{N}$, for any integer $i \geq t$, gives a natural homomorphism

$$\partial^i(M, \mathbf{N}) : \mathrm{H}_{-i+1}\mathrm{Hom}_R(\Sigma^{-1}G, N'') \to \mathrm{H}_{-i}\mathrm{Hom}_R(\Sigma^{-1}G, N),$$

of cohomology groups, where G is introduced in 4.4. Assume that $n = t$. It follows from Construction 4.4 that there is a natural isomorphism $\theta_t(M, N'') : \mathrm{H}_{-t}\mathrm{Hom}_R(P, N'') \to \mathrm{H}_{-t+1}\mathrm{Hom}_R(\Sigma^{-1}G, N'')$. Moreover, by Theorem 4.6, there is a natural isomorphism $\rho(M, N) : \mathrm{H}_{-t}\mathrm{Hom}_R(\Sigma^{-1}G, N) \to \mathrm{Ext}^{t+1}_{\mathcal{GP}}(M, N)$. Set $\partial^t_{\mathcal{GP}}(M, \mathbf{N}) := \rho(M, N) \circ \partial^t(M, \mathbf{N}) \circ \theta_t(M, N'')$. Now consider the case $n = t+1$. It follows from Proposition 3.3.4 that there is an isomorphism of cohomology groups

$$\psi_{t+1}(M, N) : \mathrm{H}_{-t-1}\mathrm{Hom}_R(\Sigma^{-1}G, N) \to \mathrm{X}^{t+2}(M, N).$$

Set $\partial^{t+1}_{\mathcal{GP}}(M, \mathbf{N}) := \psi_{t+1}(M, N) \circ \partial^{t+1}(M, \mathbf{N}) \circ \rho(M, N'')^{-1}$. Since the maps ρ, θ and ψ all are natural isomorphisms and ∂^t and ∂^{t+1} are connecting homomorphisms, $\partial^t_{\mathcal{GP}}$ and $\partial^{t+1}_{\mathcal{GP}}$ are both natural in both arguments. Moreover, the following sequence is easily seen to be exact:

$$\mathrm{Ext}^j_{\mathcal{GP}}(M, N') \longrightarrow \mathrm{Ext}^j_{\mathcal{GP}}(M, N'') \xrightarrow{\partial^j_{\mathcal{GP}}(M,\mathbf{N})} \mathrm{Ext}^{j+1}_{\mathcal{GP}}(M, N) \longrightarrow \mathrm{Ext}^{j+1}_{\mathcal{GP}}(M, N'),$$

for $j = t$ and $j = t+1$. $\square$

Proposition 4.11. *Let $\mathbf{M} = 0 \to M \to M' \to M'' \to 0$ be an exact sequence of complexes in $\widetilde{\mathcal{GP}}(R)_t$ for some $t \in \mathbb{Z} \cup \{-\infty\}$. Assume that $\mathrm{pd}_R\, M < \infty$. Then for each R-module N and each integer n, there exists a natural in $\mathbf{M}$ and N homomorphism $\partial^n_{\mathcal{GP}}(\mathbf{M}, N)$ such that the following sequence is exact:*

$$\begin{aligned} \cdots \longrightarrow \mathrm{Ext}^n_{\mathcal{GP}}(M'', N) \longrightarrow \mathrm{Ext}^n_{\mathcal{GP}}(M', N) \longrightarrow \mathrm{Ext}^n_{\mathcal{GP}}(M, N) \\ \xrightarrow{\partial^n_{\mathcal{GP}}(\mathbf{M},N)} \mathrm{Ext}^{n+1}_{\mathcal{GP}}(M'', N) \longrightarrow \cdots. \end{aligned}$$

Proof. If M and hence the other two complexes are not homologically bounded below, i.e. if $t = -\infty$, the result follows from the long exact sequence of X_R. So assume that $t \in \mathbb{Z}$. The result trivially holds for $n < t$. Also by the long exact sequence of X_R, we have the above mentioned long exact sequence for $n > t+1$. So we should prove the existence of such an exact sequence, for $n = t$ and $n = t+1$. In view of [20, 4.7], there exists a short exact sequence of complexes

$$0 \to \Sigma^{-1}G \to \Sigma^{-1}G' \to \Sigma^{-1}G'' \to 0,$$

where G, G' and G'' are related to M, M' and M'', respectively and are obtained from the Construction 4.4. Since $\mathrm{pd}_R\, M < \infty$, it follows that $\mathrm{pd}_R \Sigma^{-1}G$ is also finite and so this short exact sequence of complexes splits in each degree, because the terms of $\Sigma^{-1}G''$ are Gorenstein projective; see [6, 2.2]. Applying the functor $\mathrm{Hom}_R(\ , N)$ on this split short exact sequence, gives for any integer $i \geq t$ a natural homomorphism

$$\partial^i(\mathbf{M}, N) : \mathrm{H}_{-i+1}\mathrm{Hom}_R(\Sigma^{-1}G, N) \to \mathrm{H}_{-i}\mathrm{Hom}_R(\Sigma^{-1}G'', N).$$

Now consider the case $n = t$. As we saw in the proof of the previous theorem, there is a natural isomorphism of cohomology groups $\theta_t(M, N) : \mathrm{H}_{-t}\mathrm{Hom}_R(P, N) \to \mathrm{H}_{-t+1}\mathrm{Hom}_R(\Sigma^{-1}G, N)$. Moreover, by Theorem 4.6, there is an isomorphism $\rho(M'', N) : \mathrm{H}_{-t}\mathrm{Hom}_R(\Sigma^{-1}G'', N) \to \mathrm{Ext}^{t+1}_{\mathcal{GP}}(M'', N)$, which is natural in both arguments. Set $\partial^t_{\mathcal{GP}}(\mathbf{M}, N) := \rho(M'', N) \circ \partial^t(\mathbf{M}, N) \circ \theta_t(M, N)$. Now assume that $n = t+1$. It follows from Proposition 3.3.4, that there is a natural isomorphism $\psi_{t+1}(M'', N) : \mathrm{H}_{-t-1}\mathrm{Hom}_R(\Sigma^{-1}G'', N) \to \mathrm{X}^{t+2}(M'', N)$. So we set $\partial^{t+1}_{\mathcal{GP}}(\mathbf{M}, N) := \psi_{t+1}(M'', N) \circ \partial^{t+1}(\mathbf{M}, N) \circ \rho(M, N)^{-1}$. Since the maps ρ, θ and ψ are natural isomorphisms and ∂^t and ∂^{t+1} are connecting homomorphisms, $\partial^t_{\mathcal{GP}}$ and $\partial^{t+1}_{\mathcal{GP}}$ are natural in both arguments and the induced sequences are exact. $\square$

By applying the functor $\mathrm{Hom}_R(\ , N)$ to the split short exact sequence $0 \to \Sigma^{-1}G \xrightarrow{\chi^\flat} T^\flat \xrightarrow{\tau^\flat} P \to 0$, obtained in Construction 4.4, and then identifying the groups and maps appearing in its cohomology exact sequence, one gets the following theorem. We leave the details to the reader. For an analogue in the module case see [6, 7.1].

Theorem 4.12. *Let M be a complex with* $\inf \mathrm{H}(M) = t > -\infty$ *and* $\mathrm{Gpd}_R\, M = g < \infty$. *Then for each $i \in \mathbb{Z}$ and each R-module N, there exists a long exact sequence of cohomology groups*

$$\begin{array}{llllll} 0 \longrightarrow & \mathrm{Ext}^{t+1}_{\mathcal{GP}}(M,N) & \longrightarrow & \mathrm{Ext}^{t+1}_R(M,N) & \longrightarrow & \widehat{\mathrm{Ext}}^{t+1}_R(M,N) \\ \longrightarrow & \mathrm{Ext}^{t+2}_{\mathcal{GP}}(M,N) & \longrightarrow & \mathrm{Ext}^{t+2}_R(M,N) & \longrightarrow & \cdots \\ \longrightarrow & \cdots & \longrightarrow & \mathrm{Ext}^{g}_R(M,N) & \longrightarrow & \widehat{\mathrm{Ext}}^{g}_R(M,N) \longrightarrow 0. \end{array}$$

Remark 4.13. Similarly to what we did in this section, it is possible to consider a homologically bounded above complex N of finite G-injective dimension. In this case, one can define a relative Ext, denoted $\mathrm{ext}^*_{\mathcal{GI}}(\ , N)$, based on the class of complexes of finite Gorenstein injective dimension. All the results in this section can be rewritten for these cohomology groups.

5. Some numerical invariants for complexes

Throughout this section, R will be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Our aim in this section is to introduce and study an analogue of *Auslander's delta invariants* and *relative* and *Tate* versions of *Betti numbers* for complexes of finite homology.

5.1. ξ-invariants for complexes

The notion of Auslander's delta invariant for a finite module M over a Gorenstein complete local ring R, denoted as $\delta_R(M)$, is defined by Auslander. A maximal Cohen–Macaulay approximation (mCM) of M is a short exact sequence $0 \to Y_M \to X_M \xrightarrow{f} M \to 0$, where the middle term is mCM and the kernel of f is of finite projective dimension (see [2,14] for details on mCM approximations). If f can only be factored through itself by way of an automorphism of X_M, the approximation is called minimal. Suppose that f is a minimal approximation. Let $X_M = \underline{X}_M \oplus F$, where $\underline{X}_M$ has no free summands and F is free. Then $\delta_R(M)$ is defined as the rank of F. For any $i > 0$, $\delta^i_R(M)$ is inductively defined as $\delta_R(\Omega^i M)$, where $\Omega^i_R(M)$ denotes the ith syzygy module in a minimal free resolution of M over R. For more details see e.g. [3, Sec. 5]. Using Vogel cohomology, Martsinkovsky [17, 2.2] generalized these invariants to arbitrary commutative noetherian local rings as the dimension of the k-vector spaces $\mathrm{Ker}\,(\tilde{\varepsilon}^n_R(M,k) : \mathrm{Ext}^n_R(M,k) \to \widetilde{\mathrm{Ext}}^n_R(M,k))$. He denoted them by $\xi^i(M)$. In this subsection, we apply the same definition as in [17, 2.2] to assign new invariants to any complex M of finite homology.

Definition 5.1.1. Let M be a homologically finite complex. We define the i's ξ-invariant of M, denoted $\xi^i_R(M)$, to be

$$\xi^i_R(M) := \dim_k(\mathrm{Ker}\,(\tilde{\varepsilon}^i_R(M,k) : \mathrm{Ext}^i_R(M,k) \to \widetilde{\mathrm{Ext}}^i_R(M,k))).$$

It follows directly from the definition that these invariants coincide with Auslander delta invariants when we consider M to be a finitely generated module over a Gorenstein local ring $(R, \mathfrak{m})$.

As usual the nth syzygy of a complex M, denoted as $\Omega_n(M)$, is defined to be $C_n(P)$, where $P \to M$ is a semiprojective resolution of M. It follows from Shanuel's Lemma for complexes (see for instance [20, 1.3.6]) that $\Omega_n(M)$ is defined uniquely up to a projective direct summand. The proof of the next theorem is the same as in [17, p. 2], where a similar result is proved for modules.

Theorem 5.1.2. *Let $(R, \mathfrak{m})$ be a (commutative noetherian) Gorenstein local ring and M be a homologically bounded above complex which is homologically finite. Then, for any integer i, $\xi^i_R(M) = \delta(\Omega^i(M))$.*

Proof. Since R is Gorenstein and M is homologically bounded above, by Proposition 3.3.1 the G-projective dimension of M is finite, say g. Let $T \xrightarrow{\tau} P \xrightarrow{\pi} M$ be a surjective complete resolution of M with $\tau_i : T_i \to P_i$ bijective for all $i \geq g$. Then $\xi^i_R(M)$ equals the minimal number of generators of $P_i/\mathrm{Im}\,\tau_i$. So it can also be computed using either $\dim_k \operatorname{Coker}(\tau_i \otimes_R k)$ or $\dim_k \operatorname{Ker}(\operatorname{Hom}_R(\tau_i, k))$. Hence, the ξ-invariant measures the difference between the Tate and absolute cohomology. That is

$$\xi_R^i(M) = \dim_k(\operatorname{Ker}(\tilde{\varepsilon}_R^i(M,k) : \operatorname{Ext}_R^i(M,k) \to \widehat{\operatorname{Ext}}_R^i(M,k))).$$

But since $\operatorname{Gpd}_R M$ is finite, by 3.1.7, complete cohomology coincides with Tate cohomology. The result hence follows. □

Proposition 5.1.3. *Let R be a (not necessarily commutative and noetherian) ring with the property that* $\operatorname{spli}(R) = \operatorname{silp}(R) < \infty$. *Let M be a homologically bounded above complex. Then for any R-module N and any integer n,*

$$\operatorname{Ker} \tilde{\varepsilon}_R^n(M,N) \cong \operatorname{Ker} \tilde{\gamma}_R^n(M,N),$$

where $\tilde{\gamma}_R^n(M,N) : \operatorname{Ext}_R^n(M,N) \to \widetilde{\operatorname{ext}}_R^n(M,N)$ is introduced in 3.2.

Proof. We use reverse induction on n to prove the result. By Proposition 3.3.1, $\operatorname{Gpd}_R M < \infty$. Let $n > \operatorname{Gpd}_R M$. So $\mathrm{X}_R^n(M,N) = 0$. The balance of Theorem 3.3.9 shows that $\mathrm{x}_R^n(M,N) = 0$ for $n > \operatorname{Gpd}_R M$ and hence the result follows for these n's from the long exact sequence of cohomology groups. Assume that $n \leq \operatorname{Gpd}_R M$ and we have already proved the result for $n+1$. Consider the short exact sequence

$$0 \to L \to P \to N \to 0,$$

in which P is projective. This induces the following commutative diagrams:

$$\begin{array}{ccc}
\operatorname{Ext}_R^n(M,N) & \xrightarrow{\partial_R^n(M,N)} & \operatorname{Ext}_R^{n+1}(M,N) \\
{\scriptstyle \tilde{\varepsilon}_R^n(M,N)}\downarrow & & {\scriptstyle \tilde{\varepsilon}_R^{n+1}(M,N)}\downarrow \\
\widetilde{\operatorname{Ext}}_R^n(M,N) & \xrightarrow{\cong} & \widetilde{\operatorname{Ext}}_R^{n+1}(M,L)
\end{array}
\qquad
\begin{array}{ccc}
\operatorname{Ext}_R^n(M,N) & \xrightarrow{\partial_R^n(M,N)} & \operatorname{Ext}_R^{n+1}(M,N) \\
{\scriptstyle \tilde{\gamma}_R^n(M,N)}\downarrow & & {\scriptstyle \tilde{\gamma}_R^{n+1}(M,N)}\downarrow \\
\widetilde{\operatorname{ext}}_R^n(M,N) & \xrightarrow{\cong} & \widetilde{\operatorname{ext}}_R^{n+1}(M,L).
\end{array}$$

Let $x \in \operatorname{Ker} \tilde{\varepsilon}_R^n(M,N)$. So the first diagram implies that $(\tilde{\varepsilon}_R^{n+1}(M,N) \circ \partial_R^n(M,N))(x) = 0$. Since by the induction assumption $\operatorname{Ker} \tilde{\varepsilon}_R^{n+1}(M,L) \cong \operatorname{Ker} \tilde{\gamma}_R^{n+1}(M,L)$, we get $(\tilde{\gamma}_R^{n+1}(M,L) \circ \partial_R^n(M,N))(x) = 0$. The commutativity of the second diagram in conjunction with the fact that the morphism in its lower row is an isomorphism, implies that $x \in \operatorname{Ker} \tilde{\gamma}_R^n(M,N)$. So $\operatorname{Ker} \tilde{\varepsilon}_R^n(M,N) \subseteq \operatorname{Ker} \tilde{\gamma}_R^n(M,N)$. A similar argument proves the reverse inclusion. The inductive step and hence the proof is now complete. □

Theorem 5.1.2, in view of the above proposition, implies the following corollary.

Corollary 5.1.4. *Let $(R, \mathfrak{m})$ be a (commutative noetherian) Gorenstein local ring and let M be a homologically bounded above and homologically finite complex. Then*

$$\xi_R^i(M) = \delta(\Omega^i(M)) = \dim_k(\tilde{\gamma}_R^i(M,k) : \operatorname{Ker}(\operatorname{Ext}_R^i(M,k) \to \widetilde{\operatorname{ext}}_R^i(M,k))).$$

The following proposition is a complex version of [17, 2.4]. We recall that for any integer n, the nth Betti number $\beta_n^R(M)$ is defined to be $\beta_n^R(M) = \operatorname{rank}_k \operatorname{Ext}_R^n(M,k)$. In the case when M is a finitely generated R-module, it is interpreted as the rank of the nth free module in a minimal free resolution of M.

Proposition 5.1.5. *Let M be a complex with finitely generated homology modules. Then the following hold.*

(i) $0 \leq \xi_R^i(M) \leq \beta_i^R(M)$ *for all i.*
(ii) $\xi_R^i(M_1 \oplus M_2) = \xi_R^i(M_1) \oplus \xi_R^i(M_2)$ *for all i, where M_1 and M_2 are complexes with finitely generated homology modules.*
(iii) *If* $\operatorname{pd}_R M < \infty$, *then* $\xi_R^i(M) = \beta_i^R(M)$ *for all i.*
(iv) *If* $\inf \mathrm{H}(M) = \inf \mathrm{H}(N) = t$ *and $M \longrightarrow N$ is a surjective morphism of complexes, then* $\xi_R^t(M) \geq \xi_R^t(N)$.

Proof. Parts (i) and (ii) follow directly from the definition. Part (iii) follows from Proposition 3.1.3. For the fourth part, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^t_R(N,k) & \xrightarrow{\tilde{\varepsilon}^t_R(N,k)} & \widetilde{\mathrm{Ext}}^t_R(N,k) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^t_R(M,k) & \xrightarrow{\tilde{\varepsilon}^t_R(M,k)} & \widetilde{\mathrm{Ext}}^t_R(M,k). \end{array}$$

Since the first vertical map is a monomorphism, it is easy, by simple diagram chasing, to see that $\mathrm{Ker}\,\tilde{\varepsilon}^t_R(N,k) \subseteq \mathrm{Ker}\,\tilde{\varepsilon}^t_R(M,k)$. □

5.2. *Relative and Tate versions of Betti numbers*

Let M be a homologically finite complex of R-modules. It is known that the Betti numbers of M are important invariants of M. When M is of finite G-projective dimension, we are able to assign non-negative integers

$$\beta^{\mathcal{GP}}_n(M) = \mathrm{rank}_k\, \mathrm{Ext}^n_{\mathcal{GP}}(M,k) \quad \text{and} \quad \hat{\beta}^R_n(M) = \mathrm{rank}_k\, \widehat{\mathrm{Ext}}^n_R(M,k)$$

to M that we call the nth *relative* Betti number and nth *stable* Betti number of M, respectively (see [6, Sec. 9] for similar invariants in the module case). In our last result, we study the relations between these invariants with their absolute counterpart. Throughout, $(\ )^*$ denotes the dual functor $\mathrm{Hom}_R(\ ,R)$. The following theorem is a complex version of [6, 9.1].

Theorem 5.2.1. *Let $M \in \widetilde{\mathcal{GP}}(R)_t$ be a homologically finite complex of G-projective dimension g and $T \xrightarrow{\tau} P \xrightarrow{\pi} M$ be a complete resolution of it. Let $L = \mathrm{Ker}\,\tau$ and $L' = L_t \sqsupset$ be the hard right truncation of L at t. Set $C = C_t(T)$. Then:*

(i) $\hat{\beta}^R_n(M) = \hat{\beta}^R_n(T)$ *for all* $n \in \mathbb{Z}$.
(ii) $\hat{\beta}^R_n(M) = \hat{\beta}^R_{n-i}(C_i(T))$ *for all* $i, n \in \mathbb{Z}$.
(iii) *If* $\mathrm{pd}_R\, M < \infty$, *then* $\beta^{\mathcal{GP}}_n(M) = \beta^R_n(M)$ *and* $\hat{\beta}^R_n(M) = 0$ *for all* $n \in \mathbb{Z}$.
(iv) *If* $\mathrm{pd}_R\, M = \infty$, *then we have the following tables:*

n	$\cdots$	t	$t+1$	$\cdots$	$g+1$	$\cdots$
$\beta^{\mathcal{GP}}_n(M)$	0	$\beta^R_t(M)$	$\xi^{t+1}_R(M) \le \beta^R_{t+1}(M)$	$\beta^R_{n-1}(L')$	0	0

n	$\cdots$	$t-1$	t	$\cdots$	g	$\cdots$
$\hat{\beta}^R_n(M)$	$\beta^R_{-n+t-1}(C^*)$	$\beta^R_0(C^*) - \xi^0_R(C)$	$\beta^R_0(C) - \xi^0_R(C)$	$\beta^R_{n-t}(C)$	$\beta^R_g(M) - \xi^g_R(M)$	$\beta^R_n(M)$

Proof. (i) This is a direct consequence of the definition. Just note that T is a totally acyclic complex related both to M and T.

(ii) This equality follows from the fact that $\Sigma^{-i}T$ is a totally acyclic complex of the R-module $C_i(T)$.

(iii) The first equality follows from Proposition 4.9, while the second one follows from Propositions 3.1.3 and 3.1.7.

(iv) By definition, $\beta^{\mathcal{GP}}_n(M) = 0$ for $n < t$ and $n > g$ and is equal to $\beta^R_t(M)$ for $n = t$. The expression for $\beta^{\mathcal{GP}}_n(M)$ when $n = t+1$ follows from Definitions 4.1 and 5.1.1 and Proposition 5.1.5. By Theorem 4.7, $\mathrm{Ext}^n_{\mathcal{GP}}(M,k) \cong \mathrm{H}_{-n}\mathrm{Hom}_R(G,k)$. But for $t+1 < n \le \mathrm{Gpd}_R\, M$, $\mathrm{H}_{-n}\mathrm{Hom}_R(G,k) = \mathrm{H}_{-n}\mathrm{Hom}_R(\Sigma^1 L', k)$. The right hand side is equal to $\mathrm{H}_{-n}\Sigma^{-1}\mathrm{Hom}_R(L',k)$ which is equal to $\mathrm{H}_{-n+1}\mathrm{Hom}_R(L',k)$. But since L' is a bounded complex of projectives, we have $\mathrm{H}_{-n+1}\mathrm{Hom}_R(L',k) \cong \mathrm{Ext}^{n-1}_R(L',k)$. This concludes the result in this case.

For $n > g$, the expression for $\hat{\beta}_n^R(M)$ follows from the fact that for such n, $\mathrm{Ext}^n_{\mathcal{GP}}(M,k) = 0$. The epimorphism $\mathrm{Ext}^g_R(M,k) \to \widetilde{\mathrm{Ext}}^g_R(M,k) \to 0$ of the long exact sequence of Theorem 4.12 explains the expression for $\hat{\beta}_g^R(M)$. For $t < n < g$, note that $\hat{\beta}^R_{n-t}(C) = \beta^R_{n-t}(C)$.

Assume that $n = t$. In this case, by (ii), $\hat{\beta}_t^R(M) = \hat{\beta}_0^R(C)$. Now since C is Gorenstein projective, it is easy to see that the map $\mathrm{Ext}^0_R(C,k) \to \widehat{\mathrm{Ext}}^0_R(C,k)$ is an epimorphism. Hence $\hat{\beta}_t^R(M) = \hat{\beta}_0^R(C) = \beta_0^R(C) - \xi_R^0(C)$. When $n = t-1$, it follows from (ii) that $\hat{\beta}^R_{t-1}(M) = \hat{\beta}^R_{-1}(C)$. By [6, 9.1.3], $\hat{\beta}^R_{-1}(C) = \beta_0^R(C^*) - \xi_R^0(C)$. The expression for $\hat{\beta}_n^R(M)$, when $n < t-1$, follows from (ii) and [6, 9.1.3]. In fact by (ii), $\hat{\beta}_n^R(M) = \hat{\beta}^R_{n-t}(C)$. Now since $n - t < -1$ and C is Gorenstein projective, it follows from [6, 9.1.3] that $\hat{\beta}^R_{n-t}(C) = \beta^R_{-n+t-1}(C^*)$. □

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